# Beam Masking and its Smearing due to ISR-Induced Energy Diffusion

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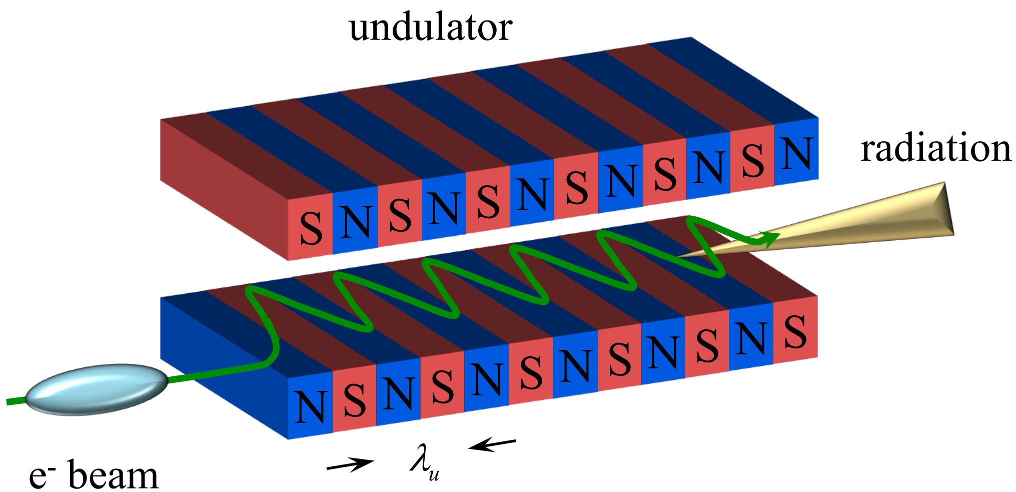
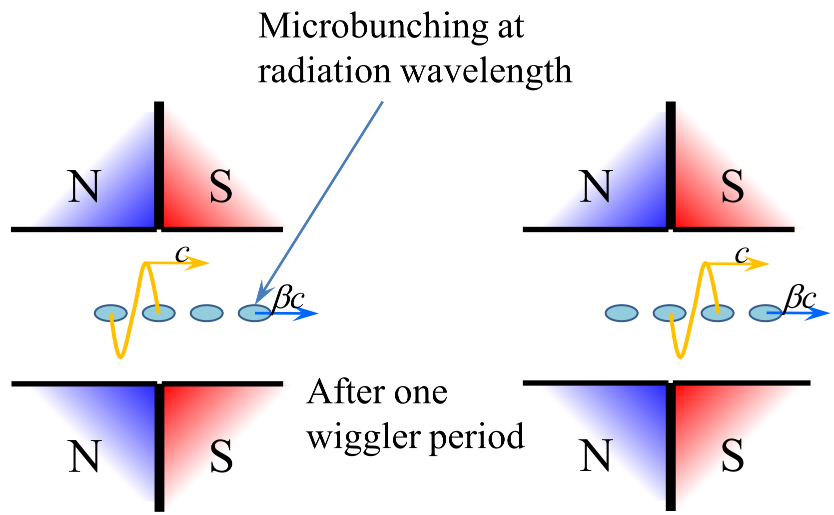
# Abstract

One of the options for increasing the longitudinal coherency of the X-ray FELs is creating the electron beam bunching at the X-ray wavelength scale. Several schemes, such as HGHG, EEHG, transverse beam masking before the Emittance Exchanger (EEX), leading to significant amplitude of the beam microbunching were recently proposed. All these scheme rely on the beam optics which include several magnetic dipoles. While the beam passes through the dipole, its energy spread increases due to quantum nature of synchrotron radiation. As a result, the bunching factor at small wavelengths reduces since electrons having different energies follow different trajectories in the bend. We study general concept of the electron beam masking and determine the beam optics which transforms the induced beam modulation into the longitudinal bunching. We rigorously calculate the reduction in the bunching factor due to incoherent synchrotron radiation (ISR) while the beam travels through the beam optics. We demonstrate that the bunching smearing in chicanes is much larger than the bunching smearing in the EEX consisting of the same bends. We determine parameters of the EEX optics which result in the smallest decrease of the electron bunching due to ISR-induced energy diffusion.

# FEL principles




Radiation slips ahead of the electron beam by one wavelength after one undulator wavelength travel distance

## Coherency of SASE light

Full transverse coherence if $L>Z_R$ or $L>\beta_x$ to provide transverse mixing between light and electron beam

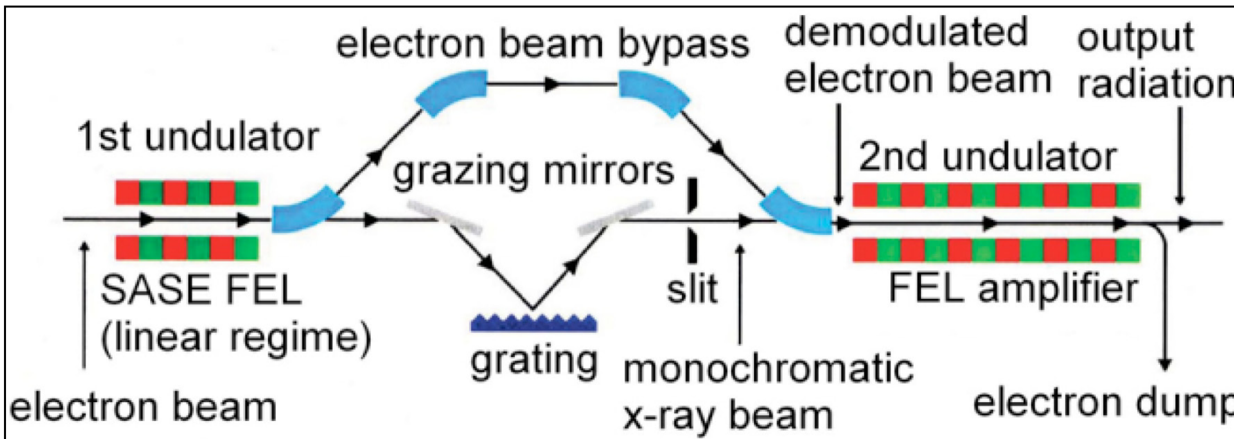
$\Delta\omega/\omega \sim \rho$ due to limited longitudinal mixing between light and electron beam

$$\boxed{\lambda_{X-ray} = \frac{\lambda_u\left(1+K^2/2\right)}{2\gamma^2}}$$

# Seeding schemes

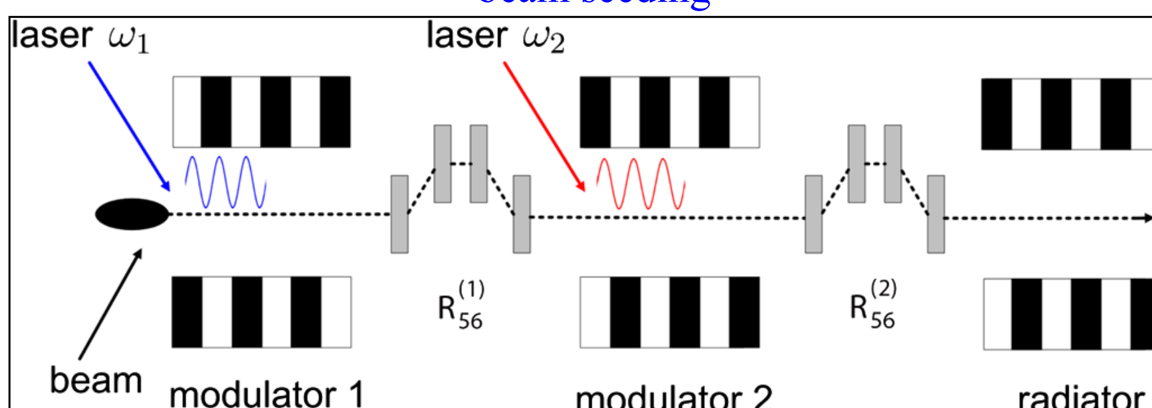
FEL mode couples electron bunching and radiation. Therefore, FEL can be seeded either by the coherent radiation or by beam bunching at the resonant wavelength.

optical seeding




J. Feldhaus et al.,
Opt. Comm. 140, 341 (1997).

beam seeding

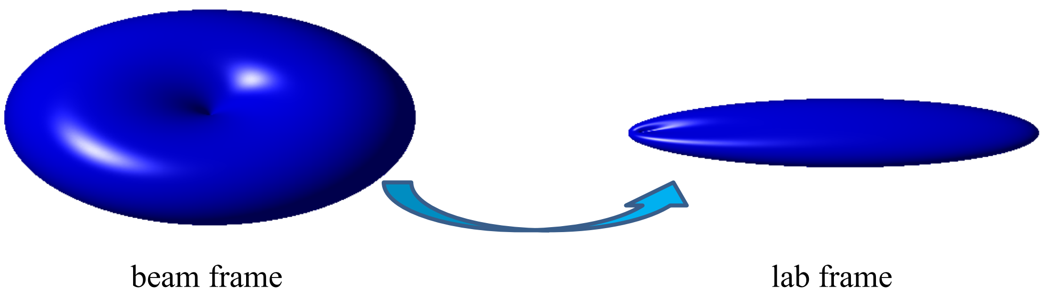


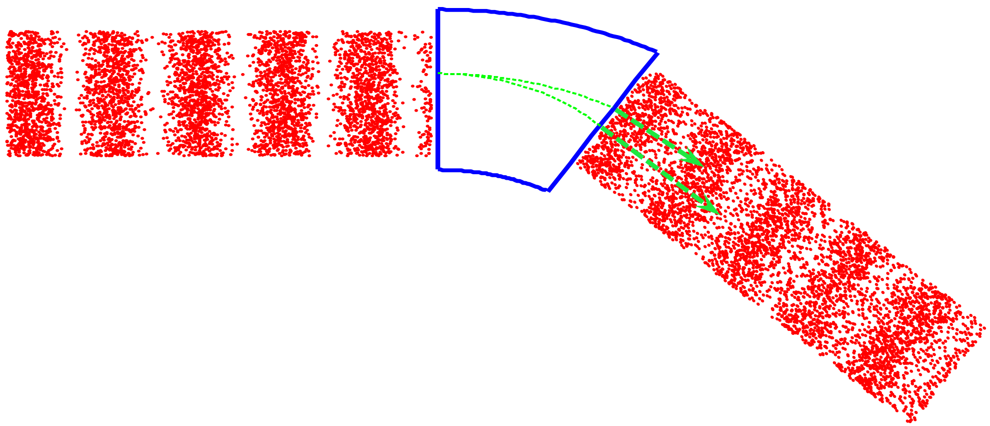

D. Xiang and G. Stupakov,
Phys. Rev. Lett. **12**, 030702 (2009).

# Mechanism for bunching degradation

Electrons emit photons in quanta which frequency (and quanta energy) strongly depends on the angle of photon emission due to Doppler shift.

Electron radiated power



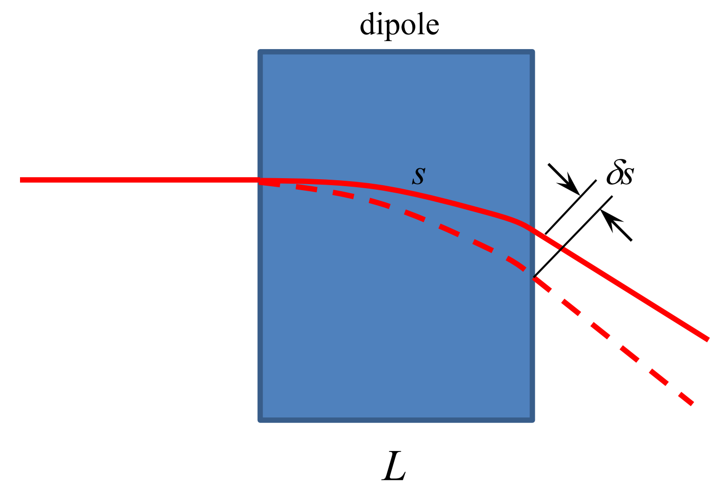



Two electrons with close 6D position in the phase space may end up at different positions after passing the bend. As a result, bunching degrades.

# Qualitative estimate

Energy diffusion due to quantum nature of incoherent synchrotron radiation

$$\frac{\delta\gamma}{\gamma} = \sqrt{\frac{55}{48\sqrt{3}}}\sqrt{\frac{\hbar e^5 B^3}{4\pi\varepsilon_0 m^5 c^6}}\,\gamma\, s^{1/2}$$




Electrons having different energies travel different distance in the bend

$$L = \rho\sin\alpha$$
$$s = \rho\alpha$$
$$\rho = \frac{\gamma mc}{eB}$$

$$s = \rho\arcsin\frac{L}{\rho} \cong L\left(1+\frac{1}{6}\left(\frac{L}{\rho}\right)^2\right)$$

$$\delta s = \frac{\partial s}{\partial\rho}\delta\rho = -\frac{1}{3}\alpha^2\frac{\delta\gamma}{\gamma}$$

Electron bunching does not degrade if

$$\delta s < \lambda_{X-ray} \qquad 3.37\alpha^{7/2}[\deg]\left(\frac{E}{10GeV}\right)^{5/2} < \lambda_{X-ray}\left[\mathring{A}\right]$$

# Vlasov Equation

## Vlasov equation in Beam Physics

$$\frac{d\vec{R}}{ds} = P(s)\vec{R}$$

Electron coordinates in 6D phase space change linearly under linear forces applied

differential transform matrix

### formal solution

$$\vec{R}(s) \equiv \exp\left(\int_{s_0}^{s} P(s')ds'\right)\vec{R}(s_0) \quad \Leftrightarrow \quad \begin{aligned} &\vec{R}(s) = M(s_0 \to s)\vec{R}(s_0) \\ &M(s_0 \to s) = \exp\left(\int_{s_0}^{s} P(s')ds'\right) \end{aligned}$$

conventional transform matrix

### constructing Vlasov equation

$$\partial_t f + \vec{V}\cdot\nabla_{\vec{R}} f = 0 \qquad \dot{\vec{R}} = \vec{V}$$

$$\boxed{\partial_s f + \nabla_{\vec{R}} f \cdot P\vec{R} = 0}$$

## conventional Vlasov equation in Plasma Physics

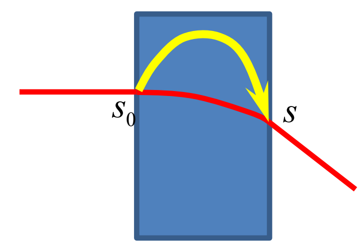
$$\partial_t f + \vec{v}\cdot\nabla f + \vec{F}\cdot\partial_{\vec{p}} f = 0$$

$$\frac{dt}{1} = \frac{dx_i}{v_i} = \frac{dp_i}{F_i} \quad \Rightarrow \quad \dot{\vec{x}} = \vec{v}, \quad \dot{\vec{p}} = \vec{F}$$

Newton equation

### Example: bend

dipole




$$P_{Bend} = \begin{pmatrix} 0 & \rho & 0 & 0 & 0 & 0 \\ -1/\rho & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & \rho & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$M_{Bend}(\alpha) = e^{\int P d\alpha} = \begin{pmatrix} \cos\alpha & \rho\sin\alpha & 0 & 0 & 0 & \rho(1-\cos\alpha) \\ -\sin\alpha/\rho & \cos\alpha & 0 & 0 & 0 & \sin\alpha \\ 0 & 0 & 1 & \rho\alpha & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ -\sin\alpha & -\rho(1-\cos\alpha) & 0 & 0 & 1 & -\rho(\alpha-\sin\alpha) \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

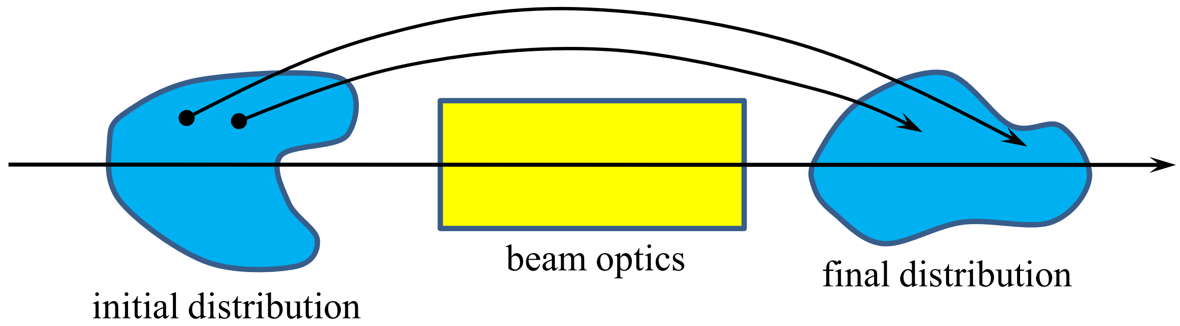
# *Transform of beam modulation*

$$\partial_s f + \nabla_{\vec{R}} f \cdot P\vec{R} = 0$$

Vlasov equation can be formally solved by tracing back the position of each electron

$$f(\vec{R},s) = f_0(M^{-1}(s_0 \to s)\vec{R}, s_0)$$

i.e. phase space density does not change along the trajectory (Liouville theorem)




beam optics

$$f_0(\vec{R}) = \bar{f}_0(\vec{R})e^{i\vec{k}_0^T\vec{R}} \qquad \Longrightarrow \qquad f(\vec{R},s) = \bar{f}_0(M^{-1}\vec{R})e^{i\vec{k}_0^T M^{-1}\vec{R}}$$

slowly changing beam envelope

monochromatic modulation

new envelope

new modulation

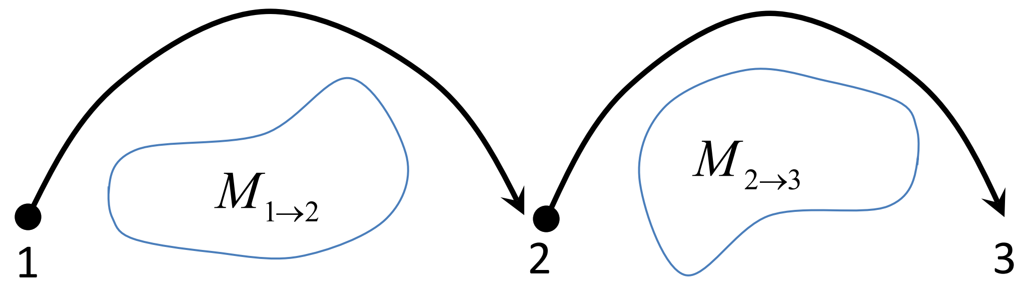
***Transforms of electron coordinates and modulation wavevector***

$$\boxed{\vec{R} = M\vec{R}_0 \qquad\qquad \vec{k} = M^{-T}\vec{k}_0}$$

# Speculative math

Transform matrix for the wavevector should depend on the beam transform matrix. It can include simple matrix operations like self matrix, inversion, and transposition

$$\vec{R} = M\vec{R}_0$$
$$\vec{k} = W\vec{k}_0$$
$$W(M) = M, M^{-1}, M^{T}, M^{-T}$$

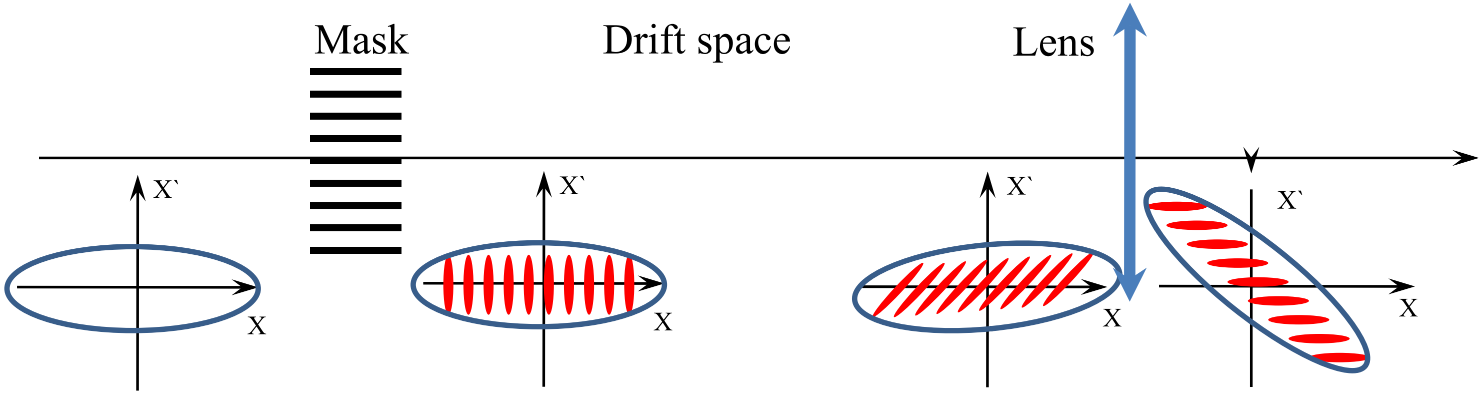



In a beamline consisting of two consecutive elements both the transform matrix for electron position and modulation wavenumber should be a product of individual transform matrices

$$M(1\to3) = M(2\to3)M(1\to2)$$
$$W(1\to3) = W(2\to3)\,W(1\to2)$$

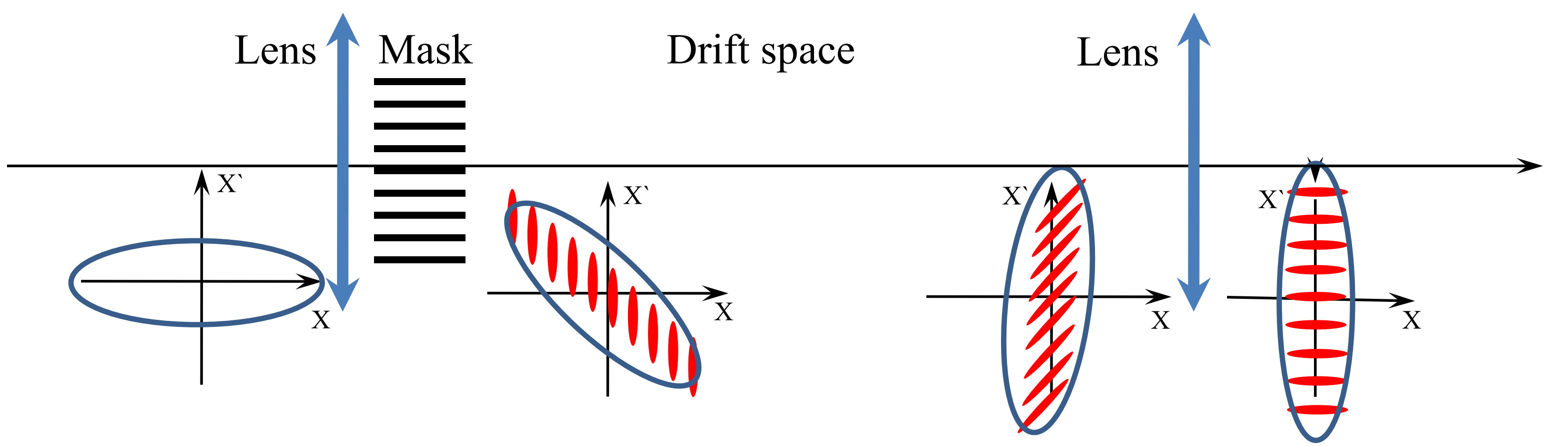
$$W(M_2M_1) = W(M_2)W(M_1) \quad \Rightarrow \quad \mathrm{W} = M \ \text{ or } \ \mathrm{W} = M^{-T}$$

# Illustration of modulation transform




scheme for creating x' modulation in the beam

Beam envelope and modulation wavevector transform differently. The beamline should be designed carefully to track evolution of both parameters simultaneously




scheme for creating x' modulation in the beam waist

# Advantages of description

*initial state*

$$f_0(\vec{R}) = \frac{1}{(2\pi)^3} \exp\left(-\frac{1}{2}\vec{R}^T \Sigma_0^{-1} \vec{R}\right) \exp\left(i\vec{k}_0^T \vec{R}\right)$$

beam envelope $\Sigma_0 = \left\langle \vec{R}^T \vec{R} \right\rangle_{f_0}$

modulation with $\vec{k}_0$

*final state*

$$f(\vec{R}) = \frac{1}{(2\pi)^3} \exp\left(-\frac{1}{2}\vec{R}^T \Sigma^{-1} \vec{R}\right) \exp\left(i\vec{k}^T \vec{R}\right)$$

beam envelope $\Sigma = \left\langle \vec{R}^T \vec{R} \right\rangle_{f}$

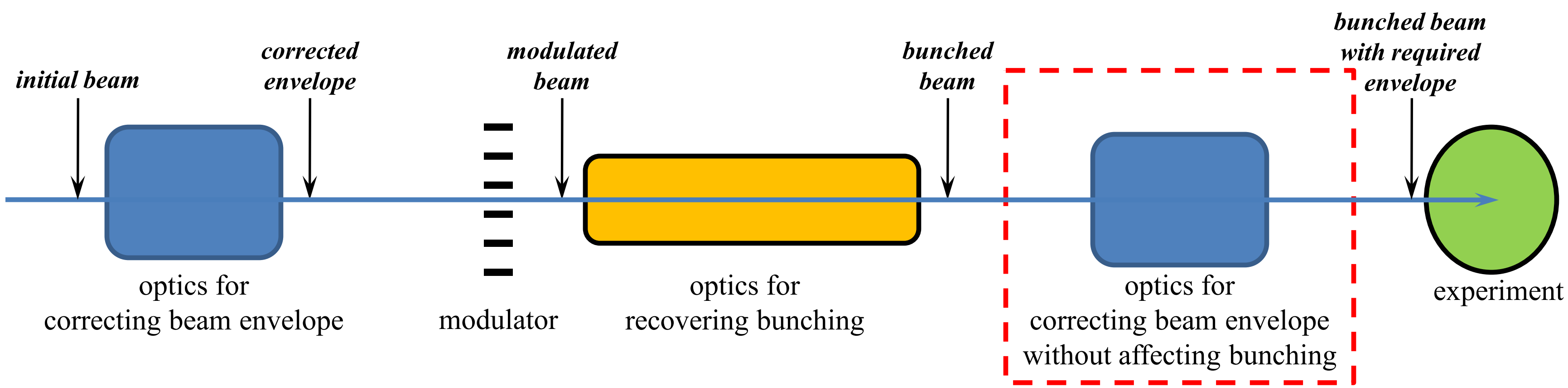
modulation with $\vec{k}$

***Transform of beam parameters***

$$\boxed{\Sigma = M\Sigma_0 M^T \qquad \vec{k} = M^{-T}\vec{k}_0}$$

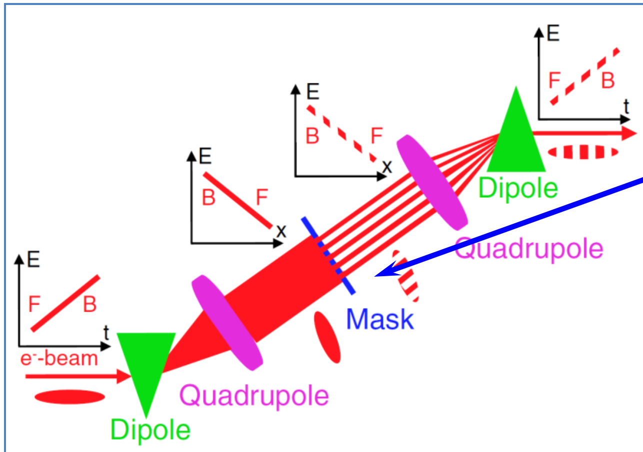
The problem generating a bunched beam with given envelope can be split into two problems:

1. One needs to determine the beamline which recovers required beam modulations
2. Placing additional beam transport upstream from the modulator section provides control over the resulting beam envelope

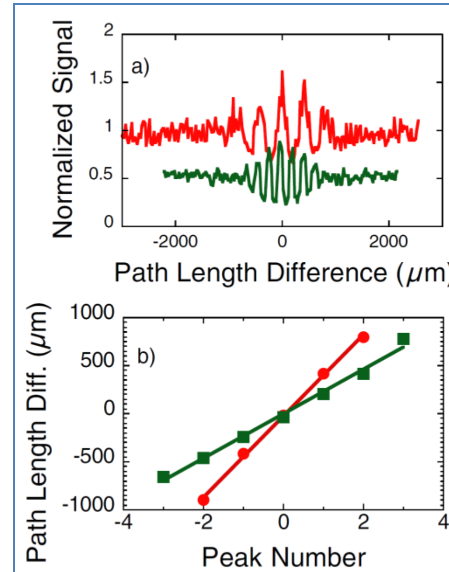


# Examples of nonoptimal optics

## Generation of trains of microbunches in Brookhaven Lab



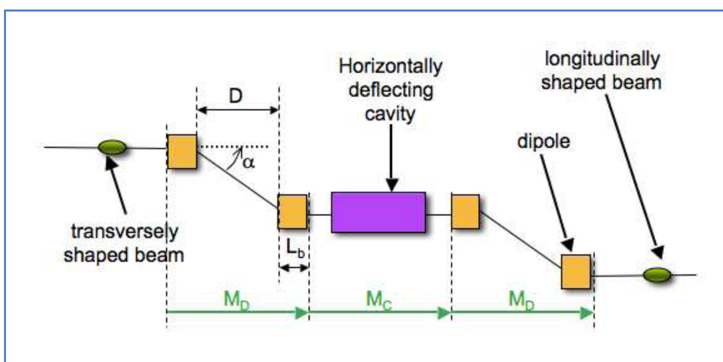

$$\sigma_\eta = \eta_x \left|\frac{\Delta\gamma}{\gamma}\right| >> \sigma_x = \sqrt{\frac{\beta_x \varepsilon_{xn}}{\gamma}}$$




P. Muggli *et al.*, Phys. Rev. Lett **101**, 054801 (2008).

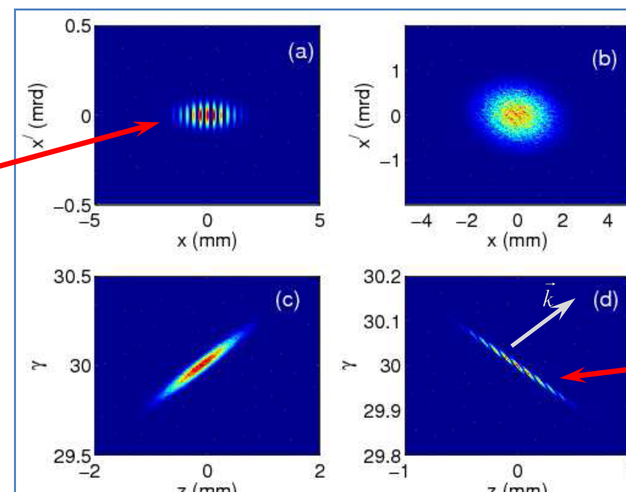
***Both schemes work ONLY for transversally bright beams***

## Emittance Exchanger study at Fermilab




Y.-E. Sun *et al.*, arXiv 2010.

$$\lambda_x \geq \sqrt{\beta_z \varepsilon_x}$$




Imperfectly transformed beam modulations

# Fokker-Planck Equation

Quantum nature of incoherent synchrotron radiation results in the energy diffusion for electrons

$\left\langle \Delta z'^2 \right\rangle^{1/2} = \sqrt{2Ds}$ introduces diffusion equation $\partial_s f = D\partial^2_{z'z'} f$

$$\partial_s f + \nabla_{\vec{R}} f \cdot P\vec{R} = D\partial^2_{z'z'} f - \cancel{\nu\partial_{z'} f}$$

diffusion term

drag term

$$D = \frac{55}{96\sqrt{3}} \frac{\hbar e^5 B^3}{4\pi\varepsilon_0 m_e^5 c^6} \gamma^2$$

Energy diffusion affects the envelope evolution much smaller than the evolution of modulation

***initial distribution function***

$$f_0(\vec{R}) = Ae^{i\vec{k}_0^T \vec{R}}$$

Fokker-Planck equation is linear, then evolution of each harmonic is independent from others

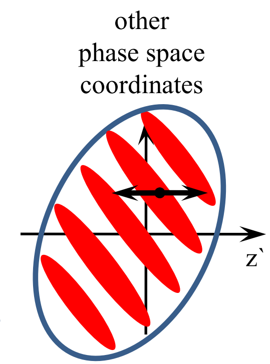
***distribution function without energy diffusion***

$$f(\vec{R},s) = Ae^{i\vec{k}_0^T M^{-1}(s,s_0)\vec{R}}$$

***distribution function with energy diffusion***

$$f(\vec{R},s) = A(s)e^{i\vec{k}_0^T M^{-1}(s,s_0)\vec{R}}$$

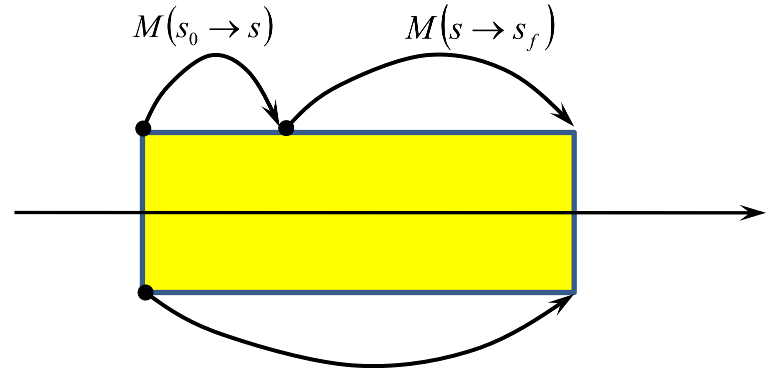
$$\frac{dA}{ds} = -D\left(\vec{k}(s)\cdot\hat{z}'\right)^2 A$$



# Symmetric beamline

***Formal solution of Fokker-Planck equation***

$$A(s) = A(s_0)\exp\left( -\int_{s_0}^{s_f} Dk_{z`}^2(s)ds \right) = A(s_0)\exp\left( -\int_{s_0}^{s_f} D\left(k_0^T M^{-1}(s_0 \to s)\hat{z}`\right)^2 ds \right)$$




$$M(s_0 \to s_f) = M(s \to s_f)M(s_0 \to s)$$

***Expressing solution through final modulation***

$$A(s) = A(s_0)\exp\left( -\int_{s_0}^{s_f} D\left(k_f^T M^{-1}(s \to s_f)\hat{z}`\right)^2 ds \right)$$

$$\vec{k}_f = M(s_0 \to s_f)\vec{k}_0$$

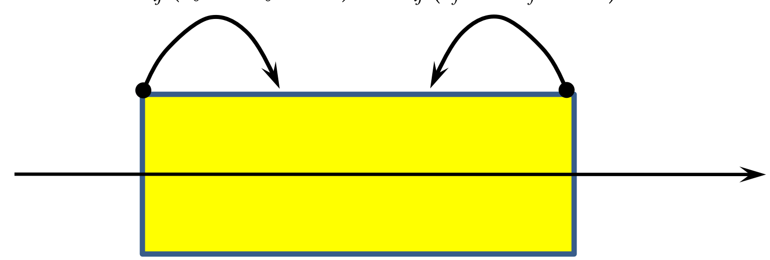
$$M(s_1 \to s_2) = \exp\left( \int_{s_1}^{s_2} P(s')ds' \right)$$

$$M^{-1}(s \to s_f) = M(s_f \to s)$$

$$\boxed{A(s) = A(s_0)\exp\left( -\int_{s_0}^{s_f} D\left(k_f^T M(s_f \to s)\hat{z}`\right)^2 ds \right)}$$

***Symmetric beamline, such as: dipole, chicane, EEX, etc.***

$$M_{ij}^2(s_0 \to s_0 + \Delta s) = M_{ij}^2(s_f \to s_f - \Delta s)$$



$$A(s) = A(s_0)\exp\left( -\int_{s_0}^{s_f} D\left(k_f^T M(s_0 \to s)\hat{z}`\right)^2 ds \right)$$

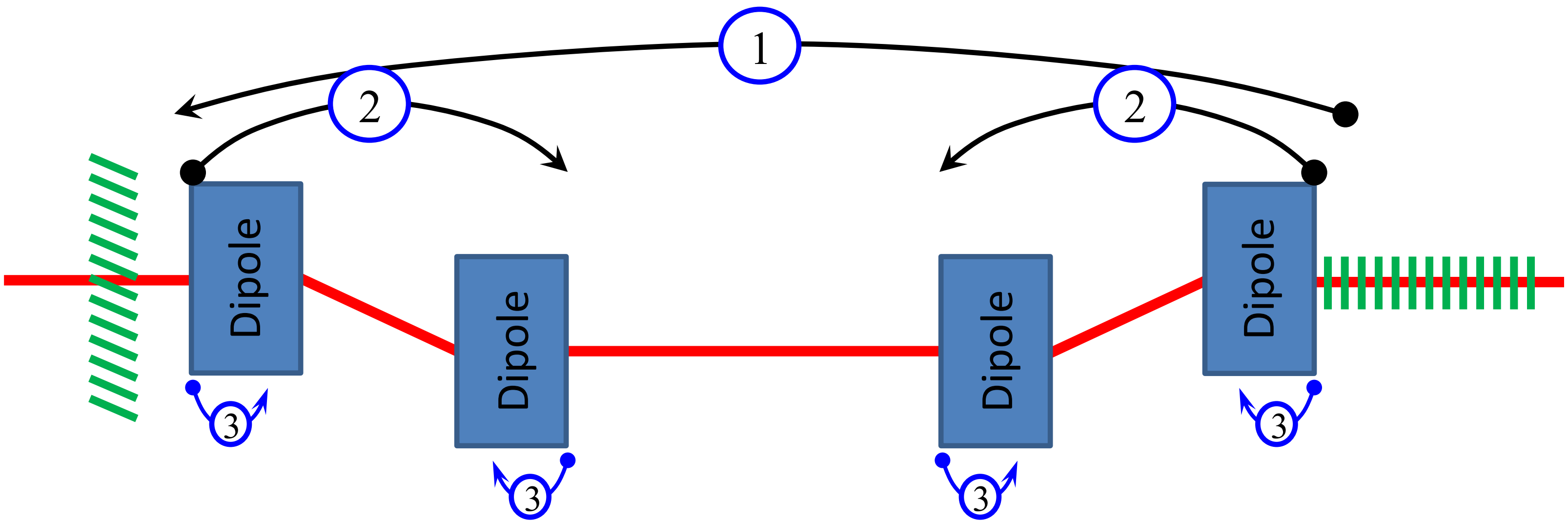
$$\vec{k}_f^T = [0, 0, 0, 0, k_{X-ray}, 0]$$

$$\boxed{A(s) = A(s_0)\exp\left( -k_{X-ray}^2 \int_{s_0}^{s_f} DM_{56}^2(s)ds \right)}$$

# Calculating modulation smearing in beamline

***Example: chicane***

$$M_{chicane} = M_{flip} M_{dogleg} M_{flip} M_{drift} M_{dogleg} = \begin{bmatrix} 1 & S+2L & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2\xi \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad M_{dogleg} = \begin{bmatrix} 1 & L & 0 & \eta \\ 0 & 1 & 0 & 0 \\ 0 & \eta & 1 & \xi \\ 0 & 0 & 0 & 1 \end{bmatrix}$$




***Algorithm for calculating bunching degradation***

1. Calculate the required bunching wavenumber at the beginning of optics $k_0 = M^T k_f$;
2. Calculate the modulation wavenumber at each dipole entrance or exit;
3. Calculate the modulation wavenumber at each point inside the dipole knowing the modulation wavenumber at its edge;
4. Integrate attenuation of modulation knowing $k_{z`}$ inside the dipole, $dA/ds = -Dk^2_{z`} \cdot A$;
5. Repeat steps 3-4 for each dipole;

$$\boxed{A = A_0 \exp\left( -2347 \left( \frac{E}{10GeV} \right)^5 \left( \frac{1\,\mathring{A}}{\lambda_{X-ray}} \right)^2 (\alpha[\deg])^7 \left[ \left( \frac{R_{56}}{\alpha^3 \rho} \right)^2 + 0.11 \right] \right)}$$

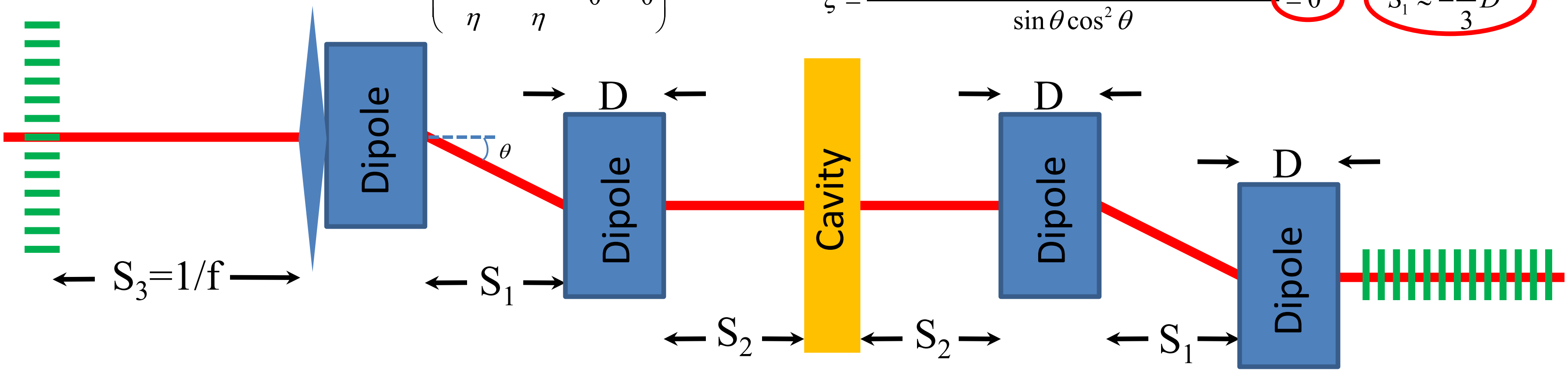
# Electron bunching through Emittance Exchanger (EEX)

$$M_{EEX} = M_{dogleg} M_{drift} M_{cavity} M_{drift} M_{dogleg} = \begin{pmatrix} 0 & 0 & -\frac{L}{\eta} & \eta \\ 0 & 0 & -\frac{1}{\eta} & 0 \\ 0 & \eta & 0 & 0 \\ -\frac{1}{\eta} & -\frac{L}{\eta} & 0 & 0 \end{pmatrix}$$

$$L = \frac{2D\cos\theta + S_1}{\cos^2\theta} + S_2$$

$$\eta = -\frac{1}{\kappa} = \frac{S_1 + 2D\cos\theta - (2D + S_1)\cos^2\theta}{\sin\theta\cos^2\theta}$$

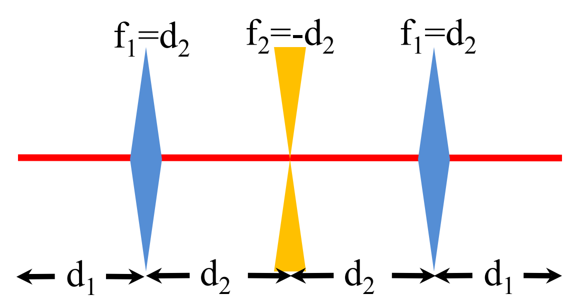
$$\xi = \frac{2D\sin\theta\cos\theta + S_1\sin^3\theta - 2D\theta\cos^2\theta}{\sin\theta\cos^2\theta} = 0 \qquad S_1 \approx -\frac{2}{3}D$$




Initial wavenumber of modulaion $\vec{k}_0 = M^T \vec{k}_f = [0, k_{x'}, 0, 0]^T$
can be created from x-modulation by placing additional
drift+lens optics before the EEX

$$\boxed{A = A_0 \exp\left( -195 \left( \frac{E}{10GeV} \right)^5 \left( \frac{1\mathring{A}}{\lambda_{X-ray}} \right)^2 (\alpha[\deg])^7 \right)}$$

## Negative drift space




$$M_{triplet}(d_1, d_2) = \begin{bmatrix} -1 & -2d_1 + 3d_2 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & \frac{4(d_1 + d_2)}{\gamma^2} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

# Comparison of bunching smearing in different beamlines

## Single bend

*qualitative estimate*

$$3.37\alpha^{7/2}[\deg]\left(\frac{E}{10GeV}\right)^{5/2} < \lambda_{X-ray}\left[\overset{\circ}{A}\right]$$

*quantitative estimate*

$$A = A_0 \exp\left(-\frac{55\pi^2}{63\cdot 96\sqrt{3}}\alpha^7\gamma^5\frac{1}{\alpha_{fine}}\left(\frac{r_e}{\lambda_{X-ray}}\right)^2\right) =$$

$$= A_0 \exp\left(-8\left(\frac{1\,\overset{\circ}{A}}{\lambda_{X-ray}}\right)^2\left(\frac{E}{10\,\mathrm{GeV}}\right)^5\alpha^7[\deg]\right),$$

## Emittance EXchanger

$$A = A_0 \exp\left(-195\left(\frac{E}{10GeV}\right)^5\left(\frac{1\overset{\circ}{A}}{\lambda_{X-ray}}\right)^2(\alpha[\deg])^7\right)$$

## Chicane

$$A = A_0 \exp\left(-2347\left(\frac{E}{10GeV}\right)^5\left(\frac{1\overset{\circ}{A}}{\lambda_{X-ray}}\right)^2(\alpha[\deg])^7\left[\left(\frac{R_{56}}{\alpha^3\rho}\right)^2 + 0.11\right]\right)$$

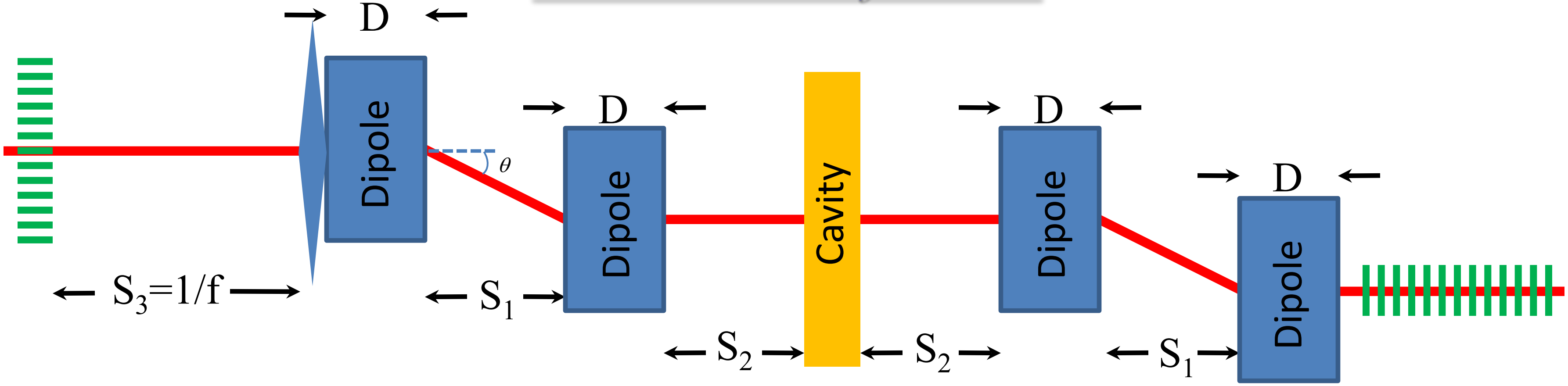
## Echo-Enabled Harmonic Generation (EEHG)

$$A = A_0 \exp\left(-1.9\cdot 10^{-3}\left(\frac{E}{10GeV}\right)^3\frac{\alpha[\deg]}{B^2[\mathrm{T}]}\frac{N_{harm}^2}{(\Delta\gamma/\gamma[0.01\%])^2}\right)$$

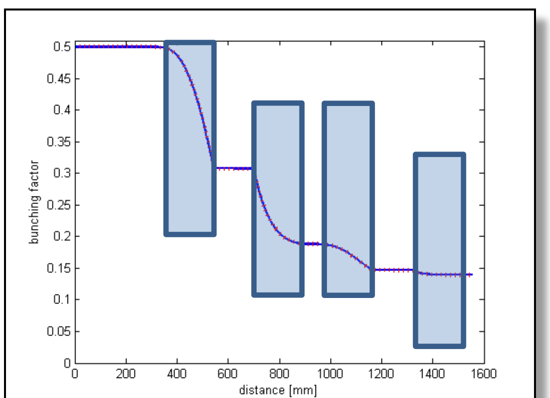
$$N_{harm}^{LCLS} = 13.6\frac{B[\mathrm{T}]\times\Delta\gamma/\gamma[0.01\%]}{\sqrt{\alpha[\deg]}} \quad @14.2\mathrm{GeV}$$ 2 nm laser

$$N_{harm}^{LCLSII} = 81.6\frac{B[\mathrm{T}]\times\Delta\gamma/\gamma[0.01\%]}{\sqrt{\alpha[\deg]}} \quad @\,4.3\mathrm{GeV}$$ 500 nm laser

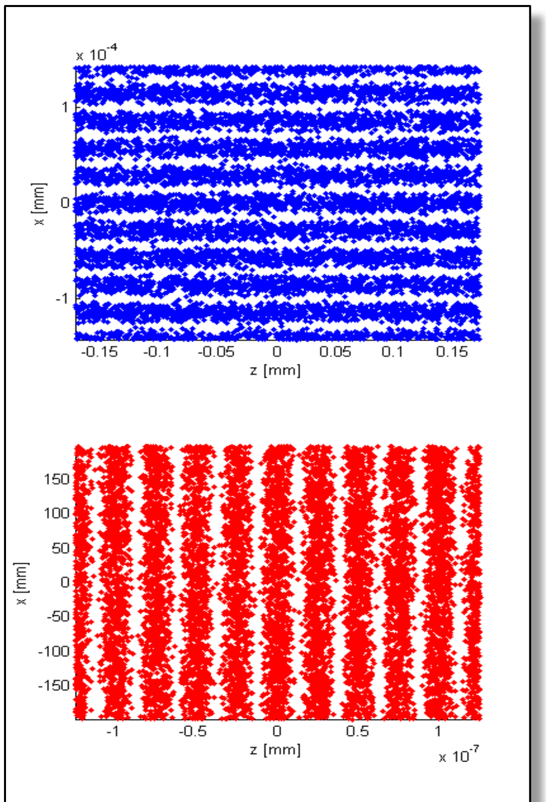
# Numerical verification




1. Generate initial ensemble of electrons with transverse density modulation;
2. Push particles through beamline elements using linear beam matrix for each element;
3. Push particles inside each bend in several steps adding random energy change on each step;




$$B = 1\,\mathrm{T}, \quad \alpha = 0.2^o, \quad E = 20\,\mathrm{GeV}, \quad \lambda_{X-ray} = 0.25\,\mathring{A}$$

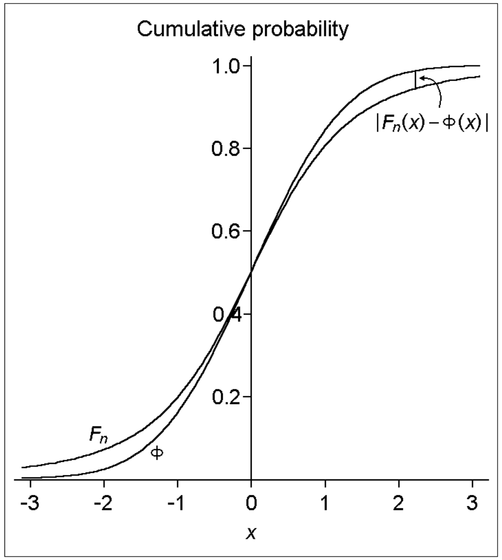


# Consistency of continuous description

***Central limit theorem***

Each electron emits photons with about 100% distribution in photon energy. Central limit theorem states that if # of photon is large, electron energy approaches Gaussian statistics, i.e. electron dynamics can be described with diffusion equation.

Berry-Esseen theorem estimates how fast the distribution function approaches Gaussian




$$Y_n = \frac{X_1 + X_2 + \cdots + X_n}{n}$$

$$F_n(x) = \int_{-\infty}^{x} Y_n(x) dx$$

$$\boxed{\left|F_n(x) - \Phi(x)\right| < \frac{0.4785\left\langle X^3 \right\rangle}{\left\langle X^2 \right\rangle^{3/2}\sqrt{n}} \sim \frac{1}{\sqrt{n}}}$$

***Bend angle radiation***

Each electron emits $\alpha_{fine}$ photons per 1/γ bend angle

Total number of emitted photons by all electrons within one bunching wavelength

$$N_{\hbar\omega} \sim \alpha_{fine} \frac{\alpha}{1/\gamma} \frac{I\lambda_{X-ray}}{ec} \sim 5.2 \cdot 10^3 \alpha[\mathrm{deg}] \frac{E}{10GeV} \frac{\lambda_{X-ray}}{1\,\mathring{A}} I[\mathrm{kA}] >> 1$$

# Results

- Formalism describing the evolution of the monochromatic beam modulations in the beamline is developed.

- Rigorous formalism describing smearing of the beam modulations due ISR-induced energy spread is developed. Straightforward algorithm for calculating the smearing effect in an arbitrary beamline is described.

- It is demonstrated that the beam bunching degradation is caused by the beamline dispersion in case of symmetric beamline.

- Smearing out of beam modulation in chicane and EEX is calculated. It is demonstrated that bunching degradation in chicane is much stronger than in EEX since chicane has nonzero dispersion.

- Analytical results are verified numerically.